

Calderón-Type Uniqueness Theorem for Stochastic Partial Differential Equations

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November 30, 2010

Abstract

In this Note, we present a Calderón-type uniqueness theorem on the Cauchy problem of stochastic partial differential equations. To this aim, we introduce the concept of stochastic pseudo-differential operators, and establish their boundedness and other fundamental properties. The proof of our uniqueness theorem is based on a new Carleman-type estimate.

1 Introduction and the main result

In his remarkable paper [1], A.-P. Calderón established a fundamental result on the uniqueness of the non-characteristic Cauchy problem for general partial differential equations. One of the main tools introduced in [1] is a preliminary version of the symbol calculation technique, which stimulated the appearance of the theory of pseudo-differential operators ([4, 5]). Later, Calderón's uniqueness theorem was extended to the operators with characteristics of high multiplicity. We refer to [6] and the references cited therein for some deep results in this topic.

In recent years, one can find more and more studies on stochastic partial differential equations (SPDEs for short). However, as far as we know, there is no work addressing the uniqueness on the Cauchy problem for general SPDEs. The main purpose of this Note is to extend the classical Calderón's uniqueness theorem (in [1]) to the stochastic setting. In order to present the key idea in the simplest way, we do not pursue the full technical generality in this Note. More precisely, we focus mainly on the Cauchy problem for SPDEs in the case of simple characteristics.

Throughout this Note, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ is a given complete filtered probability space, on which a 1-dimensional standard Brownian motion $\{w(t)\}_{t \geq 0}$ is defined. Fix $T > 0$, $n, m \in \mathbb{N} \setminus \{0\}$, and a neighborhood U of the origin in \mathbb{R}^n . Let H be a Banach space. We denote by $L_{\mathcal{F}}^{\infty}(0, T; H)$ the Banach space consisting of all H -valued $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted bounded processes, with the canonical norm; and by $L_{\mathcal{F}}^2(\Omega; C^m([0, T]; H))$ the Banach space consisting of all H -valued $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted m -th order continuous differential processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|_{C^m([0, T]; H)}^2) < \infty$, with the

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canonical norm. We consider the Cauchy problem for the following linear SPDE of order m :

$$\begin{cases} \frac{1}{i} dD_t^{m-1}u = \sum_{k=0}^{m-1} \sum_{|\alpha|=m-k} a_\alpha(t, \omega, x) D_x^\alpha D_t^k u dt \\ \quad + \sum_{|\beta|<m} \left[b_\beta(t, \omega, x) D_{t,x}^\beta u dt + c_\beta(t, \omega, x) D_{t,x}^\beta u dw(t) \right] & \text{in } (0, T) \times \Omega \times U, \\ u(0) = D_t u(0) = \cdots = D_t^{m-1} u(0) = 0 & \text{in } \Omega \times U. \end{cases} \quad (1.1)$$

In (1.1), $D_t = \frac{1}{i} \frac{\partial}{\partial t}$, $D_{x_k} = \frac{1}{i} \frac{\partial}{\partial x_k}$, i is the imaginary unit, α and β denote multi-indices, and $a_\alpha, b_\beta, c_\beta \in L^\infty_{\mathcal{F}}(\Omega; C^l([0, T] \times \bar{U}))$ for any $l \in \mathbb{N}$.

Denote by $p_m(t, \omega, x, \tau, \xi) \triangleq \tau^m - \sum_{k=0}^{m-1} \sum_{|\alpha|=m-k} a_\alpha(t, \omega, x) \xi^\alpha \tau^k$ the symbol of the principal operator appeared in (1.1), and by $\{\lambda_k(t, \omega, x, \xi); k = 1, \dots, m\}$ the characteristic roots of $p_m(t, \omega, x, \tau, \xi)$ for any $(t, \omega, x, \xi) \in (0, T) \times \Omega \times U \times \mathbb{R}^n$, i.e., $p_m(t, \omega, x, \lambda_k(t, \omega, x, \xi), \xi) = 0$. We introduce the following hypotheses:

- (H1) All roots $\lambda_k(t, \omega, x, \xi)$ ($k = 1, \dots, m$) are simple for any $(t, \omega, x, \xi) \in (0, T) \times \Omega \times U \times \mathbb{R}^n$ and $|\xi| = 1$;
- (H2) For any $(t, \omega, x, \xi) \in (0, T) \times \Omega \times U \times \mathbb{R}^n$ satisfying $|\xi| = 1$ and any complex root $\lambda_k(t, \omega, x, \xi)$, $|\operatorname{Im} \lambda_k(t, \omega, x, \xi)| \geq \varepsilon$ for some positive constant ε ;
- (H3) For any $(t, \omega, x, \xi) \in (0, T) \times \Omega \times U \times \mathbb{R}^n$ satisfying $|\xi| = 1$, and any two distinct roots $\lambda_j(t, \omega, x, \xi)$ and $\lambda_k(t, \omega, x, \xi)$, $|\lambda_j(t, \omega, x, \xi) - \lambda_k(t, \omega, x, \xi)| \geq \varepsilon$ for some positive constant ε .

The main result in this Note is stated as follows.

Theorem 1.1 *Suppose that hypotheses (H1)–(H3) hold and $u \in \bigcap_{k_1+k_2=m-1} L^2_{\mathcal{F}}(\Omega; C^{k_1}([0, T]; H^{k_2}(U)))$ is a strong solution of equation (1.1). Then there exist a neighborhood $V(\subset U)$ of the origin in \mathbb{R}^n and a sufficiently small $T' > 0$ such that u vanishes in $(0, T') \times \Omega \times V$.*

We remark that the uniqueness problem in the above simple case is by no means easy to treat. Indeed, to do this we need to introduce the concept of stochastic pseudo-differential operators (SPDOs for short) and to study their main properties. It is a little surprising that the theory of stochastic pseudo-differential operators was not available in the previous literatures although a related yet different theory for random pseudo-differential operators was introduced in [2]. Clearly, the study of stochastic pseudo-differential operators has independent interest.

We refer to [3] for the details of the proofs of the results in this Note and other results in this context.

2 Stochastic pseudo-differential operators

As a crucial preliminary to prove Theorem 1.1, in this section we present some relevant notions and results on SPDOs. First, we give the definition of symbol (for SPDOs). In the sequel, $l \in \mathbb{R}$, $p \in [1, \infty]$, and G is a domain in \mathbb{R}^n .

Definition 2.1 If a complex-valued function $a \equiv a(t, \omega, x, \xi)$ satisfies: 1) $a(t, \omega, \cdot, \cdot) \in C^\infty(G \times \mathbb{R}^n)$, $\forall (t, \omega) \in [0, T] \times \Omega$; 2) $a(\cdot, \cdot, x, \xi)$ is $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted, $\forall (x, \xi) \in G \times \mathbb{R}^n$; and 3) For any two multi-indices α and β , and any compact subset $K \subseteq G$, there exists a measurable function $M_{\alpha, \beta, K}(\cdot, \cdot)$ such that $M_{\alpha, \beta, K}(\cdot, \cdot) \in L^p_{\mathcal{F}}(0, T)$ and $|\partial_\xi^\alpha \partial_x^\beta a(t, \omega, x, \xi)| \leq M_{\alpha, \beta, K}(t, \omega)(1 + |\xi|)^{l - |\alpha|}$, $\forall (t, \omega, x, \xi) \in [0, T] \times \Omega \times K \times \mathbb{R}^n$, then a is called a symbol of order (l, p) and denoted by $a \in S^l_p(G \times \mathbb{R}^n)$.

We denote $S^\infty_p(G \times \mathbb{R}^n) = \bigcup_{m \in \mathbb{R}} S^m_p(G \times \mathbb{R}^n)$, and $S^{-\infty}_p(G \times \mathbb{R}^n) = \bigcap_{m \in \mathbb{R}} S^m_p(G \times \mathbb{R}^n)$. Now, for any $a \in S^l_p(G \times \mathbb{R}^n)$, we define a SPDO A (of order (l, p)) by

$$(Au)(t, \omega, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} a(t, \omega, x, \xi) \hat{u}(t, \omega, \xi) d\xi,$$

where $\hat{u}(t, \omega, \xi) = \int_G e^{-ix \cdot \xi} u(t, \omega, x) dx$. We denote $A \in \mathcal{L}^l_p$.

Similar to the deterministic case, we define the amplitude, kernel, adjoint operator, conjugate operator and properly supported SPDO. Furthermore, we can also establish the asymptotic expansion of symbol, etc. Here we state only the boundedness result and invertibility of stochastic elliptic operators. For $s \in \mathbb{R}$, put $H^s_{comp}(G) = \{u \in H^s(G) \mid u \in \mathcal{E}'(G)\}$ and $H^s_{loc}(G) = \{u \in \mathcal{D}'(G) \mid u\psi \in H^s(G) \text{ for any } \psi \in C^\infty_0(G)\}$. We have the following boundedness result.

Theorem 2.1 If $A \in \mathcal{L}^l_\infty$, then $A : L^p_{\mathcal{F}}(0, T; H^s_{comp}(G)) \rightarrow L^p_{\mathcal{F}}(0, T; H^{s-l}_{loc}(G))$ is continuous.

In the rest of this section, we give the definition of elliptic SPDOs and the invertibility result.

Definition 2.2 If the symbol $a \in S^l_p(G \times \mathbb{R}^n)$ satisfies that for any compact subset $K \subseteq G$, there exist two positive constants C_K and R_K such that $|a(t, \omega, x, \xi)| \geq C_K(1 + |\xi|)^m$ for any $(t, \omega) \in [0, T] \times \Omega$, $x \in K$ and $\xi \in \mathbb{R}^n$ with $|\xi| \geq R_K$, then A is called to be elliptic.

Theorem 2.2 Suppose that $A \in \mathcal{L}^l_\infty$ is elliptic. Then there exist two operators $B_1, B_2 \in \mathcal{L}^{-l}_\infty$, which are the single-sided inverse for A modulo smoothing operators, namely, $B_1 A - I \in \mathcal{L}^{-\infty}_\infty$ and $A B_2 - I \in \mathcal{L}^{-\infty}_\infty$.

3 Proof of Theorem 1.1

In this section, we give a sketch of the proof of Theorem 1.1. Similar to the deterministic setting, we need to reduce (1.1) to a SPDE of order 1. For this, we denote $A_k = \sum_{|\alpha|=m-k} a_\alpha(t, \omega, x) D_x^\alpha$. Denote

by $a_k(t, \omega, x, \xi)$ the symbol of A_k . Also, we introduce a pseudo-differential operator Λ^s , whose symbol is $(1 + |\xi|^2)^{\frac{s}{2}}$. Put $M = (\Lambda^{m-1} u, D_t \Lambda^{m-2} u, \dots, D_t^{m-1} u)^\top$. Then, the first equation of (1.1) can

be rewritten as $\frac{1}{i} dM = AM dt + f dt + F dw(t)$, where $A = \begin{pmatrix} 0 & \Lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \Lambda \\ A_m \Lambda^{1-m} & A_{m-1} \Lambda^{2-m} & \cdots & A_1 \end{pmatrix}$,

$f = (0, 0, \dots, \sum_{|\beta| < m} b_\beta(t, \omega, x) D_{t,x}^\beta u)^\top$, and $F = (0, 0, \dots, \sum_{|\beta| < m} c_\beta(t, \omega, x) D_{t,x}^\beta u)^\top$.

Denote by A_0 the pseudo-differential operator with the following symbol

$$\sigma(A_0) = \begin{pmatrix} 0 & |\xi| & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & |\xi| \\ a_m |\xi|^{1-m} & a_{m-1} |\xi|^{2-m} & \cdots & a_1 \end{pmatrix}.$$

By (H1), the operator A_0 can be diagonalized. By (H2), if the element of the diagonalization matrix is $A_1(t) + iB_1(t)$, then either $B_1(t) = 0$ or $B_1(t)$ is an elliptic SPDO.

By the boundedness and invertibility of elliptic SPDOs, the desired uniqueness result follows from the following new Carleman-type estimate for the operator “ $\frac{1}{i}dz - A_1(t)zdt - iB_1(t)zdt$ ”.

Proposition 3.1 *Suppose that $z \in L^2_{\mathcal{F}}(\Omega; C([0, T]; L^2(\mathbb{R}^n)))$ is an $L^2(\mathbb{R}^n)$ -valued semimartingale and $z(0) = z(T) = 0$ a.s. Then,*

$$\begin{aligned} & \mathbb{E} \int_0^T e^{\mu(t-T)^2} |z(t)|_{L^2(\mathbb{R}^n)}^2 dt + \frac{1}{\mu} \mathbb{E} \int_0^T e^{\mu(t-T)^2} |\mu(t-T)z(t) - B_1(t)z(t)|_{L^2(\mathbb{R}^n)}^2 dt \\ & \leq \frac{4}{\mu} \text{Re} \mathbb{E} \int_0^T \int_{\mathbb{R}^n} e^{\mu(t-T)^2} \left[\frac{1}{i} dz - A_1(t)zdt - iB_1(t)zdt \right] \cdot \overline{[i\mu(t-T)z - iB_1(t)z]} dx \\ & \quad - \frac{2}{\mu} \text{Im} \mathbb{E} \int_0^T \int_{\mathbb{R}^n} e^{\mu(t-T)^2} \left[\frac{1}{i} dz - A_1(t)zdt - iB_1(t)zdt \right] \cdot \overline{(B_1(t) - B_1^*(t))z} dx \\ & \quad - 2\mathbb{E} \int_0^T \int_{\mathbb{R}^n} (t-T) e^{\mu(t-T)^2} |dz|^2 dx - \frac{2}{\mu} \text{Re} \mathbb{E} \int_0^T e^{\mu(t-T)^2} (dz, B_1(dz))_{L^2(\mathbb{R}^n), L^2(\mathbb{R}^n)} \end{aligned}$$

for sufficiently small μ^{-1} and T , where B_1^* denotes the conjugate operator of B_1 .

Acknowledgements

This work is partially supported by the NSF of China under grants 10901032 and 10831007, and by the National Basic Research Program of China (973 Program) under grant 2011CB808002.

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